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Chebyshev Approximation by Exponential–Polynomial Products

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Let n be fixed, and consider Chebyshev approximation on a compact subset of the real line containing at least $n+1$ points by functions of the form $F(A, \cdot)$,

$$F(A, x) = \exp(a_0 x)\, L(A, x), \qquad L(A, x) = \sum_{k=1}^{n} a_k x^{k-1}.$$

In the case $n = 1$, the approximation problem is that of approximation by $a_1 \exp(a_0 x)$: best approximations are characterized by alternation and are unique [3, p. 313]. Unfortunately, a standard alternating theory does not hold for $n > 1$, despite the claims of Hobby and Rice [2, p. 106]. If a standard alternating theory held, the difference of two distinct approximations would have no more than n zeros. Let $L(A, \cdot)/L(B, \cdot)$ be a best Chebyshev approximation to $\exp(x)$ on $[\alpha, \beta]$ by polynomials of degree $n-1$ divided by polynomials of degree $n-1$. By a theorem of Meinardus and Schwedt [5, p. 318], $\exp(x) - L(A, x)/L(B, x)$ has $2n-1$ alternations on $[\alpha, \beta]$, hence $\exp((a_0+1)x)\, L(B, x) - \exp(a_0 x)\, L(A, x)$ has $2n-1$ zeros on $[\alpha, \beta]$.

However there does exist a weaker alternating theory for local best approximation [1, p. 753]. Associated with a given parameter A is the tangent space $S(A)$ generated by

$$\{x \exp(a_0 x)\, L(A, x), \exp(a_0 x), x \exp(a_0 x), \ldots, x^{n-1} \exp(a_0 x)\}.$$

In the case $a_n \neq 0$, $xL(A, x)$ is of degree n and $S(A)$ is a Haar subspace of dimension $n+1$ on any nondegenerate interval $[\alpha, \beta]$. In case $a_n = 0$, $S(A)$ is a Haar subspace of dimension n on any nondegenerate interval $[\alpha, \beta]$. From the theory of Meinardus and Schwedt [3, p. 310] we obtain

THEOREM 1. *Let $S(A)$ be a Haar subspace of dimension m. If A is (locally or globally) best to f, $f - F(A, \cdot)$ alternates m times.*

From [1, p. 753] we obtain Theorem 2.

THEOREM 2. *Let $S(A)$ be a Haar subspace of dimension $n + 1$. A necessary and sufficient condition for A to be locally best is that $f - F(A, \cdot)$ alternate $n + 1$ times.*

The observant reader will notice that the case where $S(A)$ is of dimension n and $f - F(A, \cdot)$ alternates n times has not been covered: in particular it could be conjectured that A is locally best in this case. The conjecture is however false.

EXAMPLE. Let $n = 2$ and approximate $f(x) = 1 - x^2$ on $\{-1, 0, 1\}$. $f - \frac{1}{2}$ alternates twice. Consider the sequence,

$$F(A^k, x) = \exp(x/k)[\tfrac{1}{2} + 1/k^3 - x/(2k)],$$

of approximants. We have $F(A^k, 0) = \frac{1}{2} + 1/k^3$,

$$F(A^k, 1) = \tfrac{1}{2} - 1/(4k^2) + O(1/k^3),$$
$$F(A^k, -1) = \tfrac{1}{2} - 1/(4k^2) + O(1/k^3)$$

and $F(A^k, \cdot)$ is closer to f than $\frac{1}{2}$ for all k sufficiently large. Hence $\frac{1}{2}$ is not locally best.

This example shows that the theory of [1] cannot be generalized to cover A with the tangent space of A of less than maximum dimension.

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